Note on Duality

Zepeng CHEN
The HK PolyU

Date: January 11, 2023

1 Weak, Strong Duality Theorem

1.1 Duality Form

Definition 1.1 (Symmetric Duality Form)

$$\begin{array}{llll} & \textbf{\textit{Primal}} & & \textbf{\textit{Dual}} \\ \min & c^T x & \max & \lambda^T b \\ \textit{s.t.} & Ax \geq b, x \geq 0 & \textit{s.t.} & \lambda^T A \leq c^T, \lambda \geq 0 \end{array} \tag{1}$$

Definition 1.2 (Asymmetric Duality Form)

$$\begin{array}{llll} & \textbf{\textit{Primal}} & & \textbf{\textit{Dual}} \\ \min & c^T x & \max & \lambda^T b \\ \textit{s.t.} & Ax = b, x \geq 0 & \textit{s.t.} & \lambda^T A \leq c^T \end{array} \tag{2}$$

Proof [Asymmetric is a special case of Symmetric]

$$\begin{array}{llll} & \textbf{Primal} & & \textbf{Dual} \\ \min & c^T x & \max & \lambda_1^T b - \lambda_2^T b = (\lambda_1^T - \lambda_2^T) b \\ \text{s.t.} & Ax \geq b \quad (\lambda_1), x \geq 0 & \text{s.t.} & \lambda_1^T A - \lambda_2^T A = (\lambda_1^T - \lambda_2^T) A \leq c^T \\ & -Ax \geq -b \quad (\lambda_2) & & \lambda_1 \geq 0, \lambda_2 \geq 0 \end{array} \tag{3}$$

Just let $\lambda = (\lambda_1^T - \lambda_2^T)$ then we prove it. ■

Note on Table of Duality Transformation

Dual(Max)	Primal(Min)
ith const $\leq$	ith var ≥ 0
ith const $=$	ith var unrestricted
jth var $\geq$	jth const ≥ 0
jth var unrestricted	jth const $= 0$

Definition 1.3 (Duality gap)

The gap between primal objective and dual objective.

Remark[LP's Duality gap] Duality gap for LP is zero.

Example 1.1Bounded variable LP's Dual

	Primal		Primal 2		Dual		Dual 2
min	c^Tx	min	$c^Tx_1 - c^Tx_2$	max	$l^Ty_1 - u^Ty_2$	max	$l^Ty_1 - u^Ty_2$
s.t.	$l \leq x \leq u$	s.t.	$x_1 - x_2 \geq l$	s.t.	$y_1 - y_2 \leq c$	s.t.	$y_1 - y_2 = c$
			$-(x_1 - x_2) \geq -u$		$-(y_1 - y_2) \leq -c$		
			$x_1, x_2 \geq 0$		$y_1, y_2 \geq 0$		$y_1, y_2 \geq 0$

Example 1.2Primal's variant to Dual Consider the primal LP, suppose primal and dual are feasible, let λ be a known optimal solution to the dual.

$$\begin{aligned} \text{Minimize} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{Subject to} \quad & \boldsymbol{A}\boldsymbol{x} \geq \boldsymbol{b}, \quad \boldsymbol{A} : m \times n \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

1. If the kth equation of the primal is multiplied by $\mu \neq 0$, an optimal solution w to the dual of this new problem should be: On the basis of $\mu a_k \lambda_k = a_k w_k$ and $\mu c_k \lambda_k = c_k w_k$, we have $w_k = \frac{\lambda_k}{\mu}, w_{i \neq k} = \lambda_i$.
2. If we add μ times the kth equation to the rth equation, an optimal solution w to the dual of this new problem should be: On the basis of $b_k\lambda_k + b_r\lambda_r = b_k w_k + (\mu b_k + b_r)w_r$ and $a_k\lambda_k + a_r\lambda_r = a_k w_k + (\mu a_k + a_r)w_r$, we have $w_k = \lambda_k - \mu\lambda_r, w_{i \neq k} = \lambda_i$.
3. If we add μ times the kth equation to c, an optimal solution w to the dual of this new problem should be: Suppose w is the same as λ except kth element, based on $\sum_{i=1}^m a^i w_i = \sum_{i=1}^m a^i\lambda_i + a^k w_k - a^k\lambda_k \leq (c^T + \mu a^k)$, recalling that $\sum_{i=1}^m a^i\lambda_i \leq c^T$ and if $a^k(w_k - \lambda_k) = \mu a^k$, then w is feasible too. Here $w_k = \lambda_k + \mu, w_{i \neq k} = \lambda_i$. And notice that $w^Tb = \sum_{i=1, i\neq k}^m \lambda_i + (\lambda_k + \mu)b_k = (c^T + \mu a^k)x^0$, thus w is optimal too.

$$\begin{array}{llll} & \textbf{Primal} & & \textbf{Dual} \\ \min & (c^T + \mu a^k)x & \max & w^Tb \\ \text{s.t.} & Ax = b, x \geq 0 & \text{s.t.} & w^TA \leq (c^T + \mu a^k) \end{array} \tag{4}$$

4. If the RHS changes from b to b', the resulting program is infeasible or has a finite optimal feasible solution. Since the dual feasibility does not change, and the dual problem is still feasible, so the primal problem should be infeasible or finite optimal.

$$\begin{array}{llll} & \textbf{Primal} & & \textbf{Dual} \\ \min & c^Tx & \max & w^Tb' \\ \text{s.t.} & Ax = b', x \geq 0 & \text{s.t.} & w^TA \leq c^T, \lambda \geq 0 \end{array} \tag{5}$$

1.2 Clark's Theorem

Lemma 1.1 (Clark's Theorem)

Given the following primal and dual LPs, if one of them is feasible, then the feasible region for one of them is non-empty and unbounded.

$$\begin{array}{llll} & \textbf{\textit{Primal}} & & \textbf{\textit{Dual}} \\ \min & c^T x & \max & \lambda^T b \\ s.t. & Ax \geq b, x \geq 0 & s.t. & \lambda^T A \leq c^T, \lambda \geq 0 \end{array} \tag{6}$$

Remark It's important to note that the result of the theorem is that the feasible region of one of the LPs is unbounded, but it may not be the case that the LP has unbounded objective function value with the given objective function.

Proof There are three possibilities to consider.

1. The primal is infeasible and the dual is unbounded. Done!
2. The dual is infeasible and the primal is unbounded. Done!
3. Both the primal and the dual are finite optimal.

Suppose both the primal and the dual are finite optimal, let $\hat{c} = [-1, -1, \ldots, -1]$ and consider the following systems:

1. $\exists \hat{y}$ such that $A^T \hat{y} \leq \hat{c}, \hat{y} \geq 0$
2. $\exists \hat{x}$ such that $A\hat{x} \geq 0, \hat{c}^T \hat{x} < 0, \hat{x} \geq 0$

   Farkas' lemma tells us exact one of them holds.

1. If (2) holds, then $\hat{x}$ is a feasible solution to primal LP and suppose x is also feasible to primal LP too, note that $\hat{x} \neq 0$ because $\hat{c}^T \hat{x} < 0$, for any $\lambda > 0$, we have another feasible $x + \lambda \hat{x}$ for primal too. By enlarge λ, we have a unbounded feasible region for primal LP.

$$A(x + \lambda \hat{x}) = Ax + \lambda A\hat{x} \geq b + \lambda * 0 = b$$

2. If (1) holds, then $\hat{y}$ is a feasible solution to dual LP ($\hat{y}$ is not 0 since $A^T \hat{y} \leq \hat{c}$), and suppose y is also feasible to dual LP too. Similarly, we have $y + \lambda \hat{y}$ is feasible to dual LP for any $\lambda > 0$.

■

1.3 Weak Duality Theorem

Theorem 1.1 (Weak Duality Theorem)

Let x and λ be the feasible solutions to the Primal and Dual respectively. Then $\lambda^T b \leq c^T x$.

Proof By the feasible conditions $Ax \geq b$ and $\lambda^T A \leq c^T$, we have $\lambda^\top b \leqslant \lambda^\top Ax \leqslant c^\top x$. ■

Note on *In the case of LP, the dual gap is always zero, while this is not true in other optimization problem. Weak duality theorem points out the lower bound of the primary problem, $\lambda^\top b \leqslant \lambda^\top Ax \leqslant c^\top x$ as long as we can find a λ such that $\lambda^T A \leq c^T$.*

Corollary 1.1 (Equal Primal-Dual Feasible means Optimal)

If x_0 and λ_0 are feasible to the Primal and Dual respectively and if $\lambda_0^T b = c^T x_0$, then x_0 and λ_0 are optimal to their respective problems.

Proof Assume that x_0 and λ_0 are not optimal, we can find that $c^T x_0 = \lambda_0^T b < \lambda_1^T b$, and this contradicts the Weak Duality Theorem. ■

1.4 Strong Duality Theorem

Theorem 1.2 (Strong Duality Theorem for LP)

If either the Primal or the Dual has a finite optimal solution, so does the other; the corresponding values of the objective are equal. If either problem has an unbounded objective, the other problem has no feasible solution.

Corollary 1.2 (Optimal Primal to Optimal Dual)

Let the Primal problem have an optimal basic feasible solution corresponding to the basis B. Then the vector λ satisfying $\lambda^T = c_B^T B^{-1}$ is an optimal solution to the Dual. The optimal solutions to both program are equal.

Proof If we partition A as $A = (B|D)$, and assume the optimal basis is B, then $x_B = B^{-1}b$, and the optimal value is $C_B{}^\top B^{-1}b$. The reduced cost vector is $r^T = (r_B|r_D)^T$, and $r_D{}^\top = C_D{}^\top - C_B{}^\top B^{-1}D \geqslant 0$, $r_B^\top = C_B^\top - C_B^\top B^{-1}B = 0$. Thus $C_B^\top B^{-1}D \leqslant C_D T^\top$, this means that λ^T is a feasible solution for Dual. And $\lambda^\top b = C_B^\top B^{-1}b = C_B^T x_B$, by weak duality theorem we know it is optimal. ■

Note on *Farkas lemma can be used to prove strong duality theorem and can also be proved by strong duality theorem (Ali Ahmadi, 2016, Lec. 5).*

Proof [Strong Duality to Farkas Lemma] Easy to see both condition can not holds simultaneously, then we prove if not (1) then (2).

$$\begin{array}{llll} & \textbf{Primal} & & \textbf{Dual} \\ \min & 0 & \max & \lambda^T b \\ \text{s.t.} & Ax = b, x \geq 0 & \text{s.t.} & \lambda^T A \leq 0 \end{array} \tag{7}$$

Here we prove if primal is infeasible then dual is unbounded. Easy to see dual must be feasible ($\lambda = 0$), then dual is unbounded, which means there exists λ such that $\lambda^T A \leq 0, \lambda^T b > 0$. ■

1.5 Dual solution from primal simplex table

Below is an example of how to obtain the dual solution directly from the final simplex tableau of the primal.

$$
\begin{array}{llll}
 & \textbf{Primal} & & \textbf{Dual} \\
\min & -x_1 - 4x_2 - 3x_3 & \max & 4\lambda_1 + 6\lambda_2 \\
\text{s.t.} & 2x_1 + 2x_2 + x_3 \leq 4 & \text{s.t.} & 2\lambda_1 + \lambda_2 \leqslant -1 \\
 & x_1 + 2x_2 + 2x_3 \leqslant 6 & & 2\lambda_1 + 2\lambda_2 \leqslant -4 \\
 & x_1, x_2, x_3 \geqslant 0 & & \lambda_1 + 2\lambda_2 \leqslant -3 \quad \lambda_1, \lambda_2 \leqslant 0
\end{array}
\tag{8}
$$

$$
\begin{array}{ccccccc}
\frac{3}{2} & 1 & 0 & 1 & -\frac{1}{2} & 1 & \lambda_1 \\
-1 & 0 & 1 & -1 & 1 & 2 & \lambda_2 \\
2 & 0 & 0 & 1 & 1 & 10 &
\end{array}
$$

Here the 1st and 2nd row correspond to λ_1 and λ_2, the 4th and 5th column are slack variables, the 2nd and 3rd column are basic variables. By $r^T = C_D^\top - C_B^\top B^{-1} D$, we know that for s_1, we have $0 - (\lambda_1, \lambda_2)\begin{pmatrix} 1 \\ 0 \end{pmatrix} = 1 \Rightarrow \lambda_1 = -1$, and for s_2 we have $0 - (\lambda_1, \lambda_2)\begin{pmatrix} 0 \\ 1 \end{pmatrix} = 1 \Rightarrow \lambda_2 = -1$.

1.6 Homogenous form of Dual

Theorem 1.3

Primal		***Dual***	***HD***		***PD***
finite optimal	$\rightleftarrows$	*finite optimal*			
unbounded	$\overset{\rightarrow}{\not\leftarrow}$	*infeasible*			
infeasible	$\overset{\leftarrow}{\not\rightarrow}$	*unbounded*			
infeasible	$\rightarrow$		*unbounded*	$\rightarrow$	*infeasible*
			finite optimal	$\rightarrow$	*finite optimal*

(9)

We can construct the homogenous form of dual problem as follows.

$$
\begin{array}{llllllll}
 & \textbf{\textit{Primal}} & & \textbf{\textit{Dual}} & & \textbf{\textit{HD}} & & \textbf{\textit{HD's Dual}} \\
\min & c^T x & \max & \lambda^T b & \max & \lambda^T b & \min & 0 \\
\textit{s.t.} & Ax \geq b & \textit{s.t.} & \lambda^T A \leq c^T & \textit{s.t.} & \lambda^T A \leq 0 & \textit{s.t.} & Ax \geq b \\
 & x \geq 0 & & \lambda \geq 0 & & \lambda \geq 0 & & x \geq 0
\end{array}
\tag{10}
$$

*Homogenous form has a nice property: it must be feasible, e.g. $\lambda = 0$. Note that **PD** and **P** has the same feasible region, thus they have the same feasibility.*

Lemma 1.2 (Unbounded condition)

Suppose the following LP problem is feasible:

$$\begin{aligned} \text{Minimize} \quad & \boldsymbol{c}^T\boldsymbol{x} \\ \text{Subject to} \quad & \boldsymbol{Ax} \geq \boldsymbol{b}, \quad \boldsymbol{A}: m \times n \\ & \boldsymbol{x} \geq \boldsymbol{0}. \end{aligned}$$

The optimal solution approaches to $-\infty$ if and only if there exists an $\bar{x} \neq 0$ such that $\overline{\boldsymbol{x}} \geq \boldsymbol{0}, \boldsymbol{A}\overline{\boldsymbol{x}} \geq \boldsymbol{0}, \boldsymbol{c}^T\overline{\boldsymbol{x}} < \boldsymbol{0}$.

Proof

$$\begin{array}{llllllll} & \textbf{Primal} & & \textbf{Dual} & & \textbf{HP's Dual} & & \textbf{HP} \\ \min & c^Tx & \max & \lambda^Tb & \max & 0 & \min & c^Tx \\ \text{s.t.} & Ax \geq b & \text{s.t.} & \lambda^TA \leq c^T & \text{s.t.} & \lambda^TA \leq c^T & \text{s.t.} & Ax \geq 0 \\ & x \geq 0 & & \lambda \geq 0 & & \lambda \geq 0 & & x \geq 0 \end{array} \tag{11}$$

If side: since $\bar{x}$ is a feasible solution to (HP), thus (HP) is feasible (unbounded or finite). Assume (HP) is finite optimal, so thus (HP's Dual), and $c^T\bar{x} < 0$ contradicts the weak duality theorem, thus the assumption is wrong, (HP) is unbounded and (HP's Dual) is infeasible. (HP's Dual) and (Dual) share the same feasible region, then (Dual) is infeasible too. Since (Primal) is feasible, then it must be unbounded.

If side (2): Suppose there is a feasible solution x^*, then $x^* + \lambda\bar{x}$ is also feasible ($\lambda \geq 0$), and we can increase λ to infinity and the optimal value is negative infinity.

Only if side: (Primal)'s unbounded means (Dual) is infeasible and also (HP's Dual), and (HP) must be feasible (0 is a feasible solution), thus (HP) is unbounded too. It means we can find a solution, which is feasible ($\bar{x} \geq 0, A\bar{x} \geq 0$) and $c^T\bar{x} < 0$ (unbounded and min objective function). ∎

1.7 Complementary Slackness

Theorem 1.4 (Complementary Slackness– Asymmetric Form)

Let x and λ be feasible solutions for the primal and dual programs, respectively. A necessary and sufficient condition that they both be optimal solutions is that for all i

- $x_i > 0 \Rightarrow \mathbf{y}^T\boldsymbol{a}_i = c_i$
- $x_i = 0 \Leftarrow \mathbf{y}^T\boldsymbol{a}_j < c_j$

Proof Note that in both side we have $(y^TA - c^T)x = 0$. ∎

Theorem 1.5 (Complementary Slackness– Symmetric Form)

Let x and λ be feasible solutions for the primal and dual programs, respectively. A necessary and sufficient condition that they both be optimal solutions is that for all i and j (where a^j is the jth row of A)

- $x_i > 0 \Rightarrow \mathbf{y}^T\mathbf{a}_i = c_i$
- $x_i = 0 \Leftarrow \mathbf{y}^T\mathbf{a}_i < c_i$
- $\lambda_j > 0 \Rightarrow \mathbf{a}^j\mathbf{x} = b_j$
- $\lambda_j = 0 \Leftarrow \mathbf{a}^j\mathbf{x} > b_j$

Lemma 1.3 (Primal-Dual Feasible + Complementary Slackness = Optimal)

Given a primal-feasible solution x and a dual-feasible solution y, x and y are optimal iff the complementary slackness conditions hold.

Example 1.3 Consider the following LP (P. Williamson, 2014, PS. 1), assume that v_i, s_i are positive and $\frac{v_1}{s_1} \geq \frac{v_2}{s_2} \geq \cdots \geq \frac{v_n}{s_n}$, let k be the largest index such that $s_1 + s_2 + \cdots + s_{k-1} \leq B$. Find a optimal solution to primal and dual.

$$\begin{array}{llll}
 & \textbf{Primal} & & \textbf{Dual} \\
\max & \sum_{i=1}^{n} v_i x_i & \min & \sum_{i=1}^{n} y_i + By_0 \\
\text{s.t.} & \sum_{i=1}^{n} s_i x_i \leq B & \text{s.t.} & [s \quad I^T](y_0, ..., y_n)^T \geq v \\
 & x_i \leq 1 \quad i = 1, \ldots, n & & y_i \geq 0 \quad i = 0, ..., n \\
 & x_i \geq 0 \quad i = 1, \ldots, n & &
\end{array} \tag{12}$$

Solution *The logic is that the first $k-1$ variable contribute most value to objective, thus they must be the maximum value, that is, 1. And the kth variable can achieve maximum smaller than 1 due to the first constraint. By complementary slackness we know the dual variable behind kth must be 0, and solve the following equations we derive the dual solution.*

$$x_i = \begin{cases} 1 & i < k \\ \frac{B-(s_1+s_2+\cdots+s_{k-1})}{0} & i = k \\ 0 & i > k \end{cases}$$

$$y_i = \begin{cases} \frac{v_k}{s_k} & i = 0 \\ s_i\left(\frac{v_i}{s_i} - \frac{v_k}{s_k}\right) & 0 < i < k \\ 0 & i \geq k \end{cases}$$

1.8 Degeneracy and Uniqueness under Duality

1. Since λ is m-dimensional, dual degeneracy implies more than m reduced costs that are zero.
1. If dual has a nondegenerate optimal solution, the primal problem has a unique optimal solution. However, it is possible that dual has a degenerate solution and the dual has a unique optimal solution.

1.9 Redundant Equations (Luenberger and Ye, 2015, Ch. 4)

Definition 1.4 (Reduandant equations)

Corresponding to the system $Ax = b, x \geq 0$, we say the system has redundant equations if there is a nonzero λ satisfying $\lambda^T A = 0, \lambda^T b = 0$.

Remark This means that one of the equations can be expressed as a linear combination of the others.

Definition 1.5 (Null variable)

Corresponding to the system $Ax = b, x \geq 0$, a variable x_i is said to be a null variable if $x_i = 0$ in every solution.

Example 1.4

$$2x_1 + 3x_2 + 4x_3 + 4x_4 = 6$$
$$x_1 + x_2 + 2x_3 + x_4 = 3$$
$$x_1 \geqslant 0, x_2 \geqslant 0, x_3 \geqslant 0, x_4 \geqslant 0$$

Twice the second row minus the first row leads to $x_2 + 2x_4 = 0$. Thus, both of two variables are null variable.

Lemma 1.4 (Null value theorem)

If S is not empty, the variable x_i is a null variable in the system $Ax = b, x \geq 0$ iff there is a nonzero vector λ such that $\lambda^T A \geq 0, \lambda^T b = 0$ and the ith component of $\lambda^T A$ is strictly positive.

Definition 1.6 (Nonextremal variable)

A variable x_i in the system $Ax = b, x \geq 0$ is nonextremal if the inequality $x_i \geq 0$ is redundant.

Lemma 1.5 (Nonextremal variable theorem)

If S is not empty, the variable x_j is a nonextremal variable for the system $Ax = b, x \geq 0$ iff there is $\lambda \in E^m$ and $d \in E^n$ such that

$$\boldsymbol{\lambda}^T \mathbf{A} = \mathbf{d}^T \quad d_j = -1, \quad d_i \geqslant 0 \quad \text{for} \quad i \neq j$$

and such that

$$\boldsymbol{\lambda}^T \mathbf{b} = -\beta \quad \text{for some } \beta \geq 0$$

Lemma 1.6 (Inconsistent systems of linear inequalities (Bertsimas et al., 1997, P. 194))

Let $a_1, ..., a_m$ be some vectors in R^n, with $m > n + 1$. Suppose that the system of inequalities $\mathbf{a}_i'\mathbf{x} \geq b_i, i = 1, \ldots, m$, does not have any solutions. Show that we can choose $n + 1$ of these inequalities, so that the resulting system of inequalities has no solutions.

2 Dual Simplex Method

3 Shadow Price and Sensitivity Analysis

Definition 3.1 (Shadow Price)

The shadow price to a constraint i is the rate of the change in the objective function value as a result of a change in the value of b_i.

Definition 3.2 (Simplex Multiplier)

$\lambda^T = c_B^T B^{-1}$

3.1 Introducing a new variable

$$\begin{aligned} \text{Minimize} \quad & c^T x + c_{n+1} x_{n+1} \\ \text{Subject to} \quad & Ax + a_{n+1} x_{n+1} = b \\ & x \geq 0, x_{n+1} \geq 0 \end{aligned}$$

1. The feasibility of the solution is not affected (the feasibility region is enlarged), but the solution may not be optimal. We now have more choices to form the basis.
2. Firstly, check if $c_{n+1} - C_B^T B^{-1} a_{n+1} \geq 0$ still holds. If so, the former optimal solution is still optimal, else x_{n+1} should enter the basis and we need to find the leave variable.

3.2 Introducing a new constraint

Lemma 3.1 (Introducing a new constraint)

Consider the LP in standard form, assume x^0 is an optimal solution to the problem. Introducing a new constraint $\boldsymbol{a}^T \boldsymbol{x} \leq \varphi$.

$$\begin{aligned} \textit{Minimize} \quad & \boldsymbol{c}^T \boldsymbol{x} \\ \textit{subject to} \quad & \boldsymbol{A}\boldsymbol{x} = \boldsymbol{b}, \boldsymbol{x} \geq \boldsymbol{0} \end{aligned}$$

1. *Prove that if $\boldsymbol{a}^T \boldsymbol{x} \leq \varphi$, then x^0 is optimal for the new problem too.*
2. *Prove that if $\boldsymbol{a}^T \boldsymbol{x} > \varphi$, then either there exists no feasible solution to the original problem or there exists an optimal solution x^* such that $a^T x^* = \varphi$.*

Proof

$$\begin{array}{llllllll} & \textbf{P1} & & \textbf{D1} & & \textbf{P2} & & \textbf{D2} \\ \min & c^T x & \max & \lambda^T b & \min & c^T x & \max & \lambda^T b - \lambda_{m+1}\varphi \\ \text{s.t.} & Ax = b & \text{s.t.} & \lambda^T A \leq c^T & \text{s.t.} & Ax = b, a^T x \leq \varphi & \text{s.t.} & \lambda^T A - \lambda_{m+1} a \leq c^T \\ & x \geq 0 & & & & x \geq 0 & & \lambda_{m+1} \geq 0 \end{array} \tag{13}$$

If $\boldsymbol{a}^T\boldsymbol{x} \leq \varphi$, assume λ^0 is the optimal solution to (D1), then $c^Tx^0 = (\lambda^0)^Tb$. We can construct a solution $(\lambda^0, \lambda_{m+1} = 0)$ to (D2). Note that $(\lambda^0, \lambda_{m+1} = 0)$ is feasible to (D2) and $c^Tx^0 = (\lambda^0)^Tb - 0 \cdot \varphi$, thus x_0 is still the optimal solution to (P2).

$$
\begin{array}{llll}
 & \textbf{P3} & & \textbf{D3} \\
\min & 0 & \max & \lambda^T b - \lambda_{m+1}\varphi \\
\text{s.t.} & Ax = b, a^T x \leq \varphi & \text{s.t.} & \lambda^T A - \lambda_{m+1} a \leq 0 \\
 & x \geq 0 & & \lambda_{m+1} \geq 0
\end{array} \tag{14}
$$

If $\boldsymbol{a}^T\boldsymbol{x} > \varphi$, since 0 is a feasible solution to (D3), thus (D3) is feasible.

1. If (D3) is unbounded, then (P3) is infeasible. Since (P2) and (P3) share the same feasible region, thus (P2) is infeasible too.
2. If (D3) is finite optimal, so thus (P3) and (P2). Assume that for all optimal solution x^*, $a^Tx^* < \varphi$, and also we have $\lambda_{m+1} = 0$ according to the complementary slackness property. That is, (D2) and (D1) share the same feasible region and achieves the same optimal solution, (D2) can be rewritten as (D1). Thus (D2)'s dual problem should share the same optimal solution as (P1) too, and all optimal solution for (P1) should satisfy $a^Tx \leq \varphi$, and this contradicts our assumption. Therefore, there exists an optimal solution x^* for (P2), where $a^Tx^* = \varphi$.
3. If (D3) is finite optimal, so thus (P3) and (P2). Assume that for all optimal solution x^*, $a^Tx^* < \varphi$, and also we have $\lambda^*_{m+1} = 0$ according to the complementary slackness property. Denote the optimal solution set for (P1), (D1), (P2), (D2) as X_1, Y_1, X_2, Y_2, then for any $x_1 \in X_1, \lambda_1 \in Y_1, x_2 \in X_2, \lambda_2 \in Y_2$, $c^\top x^2 = \left((\lambda^1)^\top\right) b = (\lambda^2)^\top b = c^\top x'$. This means $X_2 \in X_1$. To prove this, assume there is $x_2 \in X_2$ and $x_2 \notin X_1$, but x_2 is feasible to (P1) and $c^Tx^2 = c^Tx^1$, contradiction. Note that the optimal solution set must be convex, and it means the hyperplane $a^Tx = \varphi$ splits the set X_1, and X_2 is the part of X_1 located in the negative half space of $a^Tx = \varphi$. Thus $X_2 \cap \left\{x | a^\top x = \varphi\right\}$ is not empty, there exists an optimal solution x^* such that $a^Tx^* = \varphi$.

■

$$
\begin{array}{ll}
\text{Minimize} & c^T x \\
\text{Subject to} & Ax = b \\
 & a^{n+1}x = b_{n+1} \\
 & x \geq 0
\end{array}
$$

1. The current solution is also optimal if it satisfies the augmented constraint. Introducing a new constraint is actually introducing a new hyperplane and reduce the feasibility region.
2. Otherwise, ...

3.3 Change Cost coefficient for a non-basic variable

$$\begin{aligned} \text{Minimize} \quad & \textstyle\sum_{i=1,i\neq j}^{n} c_i x_i + (c_j + \Delta) x_j \\ \text{Subject to} \quad & Ax = b \\ & x \geq 0 \end{aligned}$$

1. The feasibility $B^{-1}b$ does not change too.
2. Note that the simplex multipliers $C_B B^{-1}$ are not affected, thus the next thing is to check the reduced cost whether $r_j(\Delta) = c_j + \Delta - \lambda^T a_j = \Delta + r_j \geq 0$. If so, then the optimal remains, otherwise, this non-basic variable should enter the basis.

3.4 Change Cost coefficient for a basic variable

Consider a change in the cost coefficient c_i of a basic variable x_i to $c_i + \Delta$:

1. The feasibility holds.
2. The basic cost vector changes from c_B to $c_B(\Delta) = c_B + \Delta e_i$.
3. The updated simplex multipliers are $\lambda^T(\Delta) = (c_B(\Delta))^T B^{-1} = \lambda^T + \Delta e_i^T B^{-1}$.
4. The reduced cost coefficient for a non-basic variable x_j is $r_j(\Delta) = c_j - \lambda^T(\Delta) a_j = r_j - \Delta e_i^T B^{-1} a_j = r_j - \Delta y_{ij}$. Thus the range of Δ for which the current solution remains optimal is given by $\max_{y_{ij}<0} \frac{r_j}{y_{ij}} \leq \Delta \leq \min_{y_{ij}>0} \frac{r_j}{y_{ij}}$.

3.5 Changing RHS scalar

Consider a change in a RHS scalar b_i to $b_i + \Delta$:

1. The simplex multipliers are unaffected and the optimality condition holds.
2. If the feasibility holds, then it is still optimal. If the feasibility does not hold, then apply the Dual Simplex Method.
3. Note that $x_B(\Delta) = B^{-1}(b + \Delta e_i) = x_B + \Delta B^{-1} e_i$, thus it may not be feasible.
4. The range of Δ for which the current solution remains optimal is given by $\max_{\beta_{ki}>0} \frac{-x_{B_k}}{\beta_{ki}} \leq \Delta \leq \min_{\beta_{ki}<0} \frac{-x_{B_k}}{\beta_{ki}}$, where β_{ki} is the kith element of B^{-1}.
5. If the current solution remains optimal, the objective function value changes to $z^*(\Delta) = z^* + \Delta\lambda_i$, where λ_i is the ith element in the vector of simplex multipliers.

3.6 Changing a non-basic column

Consider a change in a coefficient a_{kj} in a non-basic column vector $a_j, k = 1, 2..., m; j = m+1, ..., n$ to $a_{kj} + \Delta$, that is, $a_j(\Delta) = a_j + \Delta e_k$.

1. The feasibility of the solution and simplex multipliers remain unaffected.
2. The reduced cost coefficient of x_j is $r_j(\Delta) = c_j - \lambda^T a_j(\Delta) = r_j - \Delta\lambda^T e_k = r_j - \Delta\lambda_k$.
3. The range of Δ for which the current solution remains optimal is $\Delta\lambda_k \leq r_j$. If $\lambda_k = 0$, then the optimality is not affected by row k.

3.7 Changing a basic column

Consider a change in a coefficient a_{ki} in a basic column vector $a_i, k = 1, 2..., m; j = m+1, ..., n$ to $a_{kj} + \Delta$.

1. The feasibility of the solution and simplex multipliers remain unaffected.
2. The updated basis is $B(\Delta) = B + \Delta e_k e_i^T = B\left(I + \Delta B^{-1} e_k e_i^T\right)$, and $B^{-1}(\Delta) = \left(I - \varphi B^{-1} e_k e_i^T\right) B^{-1}$, where $\varphi = \left[\beta_{ik} + \Delta^{-1}\right]^{-1}$.
3. The updated solution is $x_B(\Delta) = x_B - \varphi x_i^* B_{\cdot k}^{-1}$, and the condition for primal feasibility is $\max_{\{q \in B | x_i^* \beta_{qk} < 0\}} \frac{x_q^*}{x_i^* \beta_{qk}} \leq \varphi \leq \min_{\{q \in B | x_i^* \beta_{qk} > 0\}} \frac{x_q^*}{x_i^* \beta_{qk}}$.
4. The simplex multipliers is $\lambda(\Delta) = \lambda - \varphi \lambda_k e_i^T B^{-1}$, and the reduced cost is $r_N(\Delta) = r_N - \varphi \lambda_i \left(B^{-1} N\right)_k$. And the condition for dual feasibility is $\max_{\{j \in N | \lambda_i a_{kj} < 0\}} \frac{r_j}{\lambda_i a_{kj}} \leq \varphi \leq \min_{\{j \in N | \lambda_i a_{kj} > 0\}} \frac{r_j}{\lambda_i a_{kj}}$.

4 Lagrange Duality

Lagrange dual problem is always a convex optimization problem regarding the dual variable, i.e., $\min_x \left(f(x) - \lambda^\top g(x)\right)$ is concave regarding λ. On the basis of weak duality theorem, we can derive the lower bound of the primal problem.

$$
\begin{array}{llll}
 & \textbf{Primal} & \textbf{Dual} & \\
\min & f(x) & \max\limits_{\lambda \geq 0} & (\min\limits_{x} f(x) - \lambda^T g(x)) \\
\text{s.t.} & g_i(x) \geq 0, x \geq 0 & \text{where} & g(x) = (g_1(x), ..., g_n(x))^T
\end{array}
\tag{15}
$$

Note on Sign of Lagrange multiplier *Be careful to the sign of Lagrange multiplier, since* $g(x) \geq 0$ *and* $\lambda \geq 0$, *and we want to minimize the objective function. According to the logic of penalty method, the objective function should minus* $\lambda g(x)$ *to ensure that* $g(x) \geq 0$ *holds when we minimize the objective function. Thus, if* $g(x) \leq 0$, *the Lagrange function should be* $f(x) + \lambda g(x)$.

Definition 4.1 (Lagrangian function for standard LP)

Suppose the original problem is the following, call this problem (P),

$$\textit{minimize } f(x) \textit{ subject to } h(x) = b, x \in X \quad L(x, \lambda) = f(x) + \lambda^T (h(x) - b)$$

then the Lagrangian of (P) *is defined as*

$$L(x, \lambda) = f(x) - \lambda^T (h(x) - b)$$

for $\lambda \in \mathbb{R}^m$. λ *is known as the Lagrange multiplier.*

Remark Note that the sign of λ does not change our result because of the equality constraint.

Theorem 4.1 (Lagrangian sufficiency)

Let $x^* \in X$ *and* $\lambda^* \in \mathbb{R}^m$ *be such that*

$$L(x^*, \lambda^*) = \inf_{x \in X} L(x, \lambda^*) \quad \text{and} \quad h(x^*) = b$$

Then x^* *is optimal for* (P).

Proof ■

5 KKT condition

Bibliography

Ali Ahmadi, Amir (2016). *ORF 523 Convex and Conic Optimization*. Princeton University.

Bertsimas, Dimitris, John N. Tsitsiklis, and John Tsitsiklis (Feb. 1997). *Introduction to Linear Optimization*. unknown edition. Belmont, Mass: Athena Scientific. ISBN: 978-1-886529-19-9.

Luenberger, David G. and Yinyu Ye (July 2015). *Linear and Nonlinear Programming*. 4th ed. 2016 edition. New York, NY: Springer. ISBN: 978-3-319-18841-6.

P. Williamson, David (2014). *ORIE 6300 Mathematical Programming I*. Cornell University.